




## Article

# A Theoretical Model for the Development of Mathematical Talent through Mathematical Creativity

Zeidy M. Barraza-García <sup>1,\*</sup> , Avenilde Romo-Vázquez <sup>1</sup>  and Solange Roa-Fuentes <sup>2</sup> 

<sup>1</sup> Programa de Matemática Educativa, CICATA-Instituto Politécnico Nacional, CDMX 11500, Mexico; aromov@ipn.mx

<sup>2</sup> Escuela de Matemáticas, Universidad Industrial de Santander, Bucaramanga 680002, Colombia; sroa@matematicas.uis.edu.co

\* Correspondence: zeidy.barraza@gmail.com

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**Abstract:** This study was conducted from a perspective that adopts a broad vision of mathematical talent, defined as the potential that a subject manifests when confronting certain types of tasks, in a successful way, that generate creative mathematical activity. To analyse this, our study proposes a Praxeological Model of Mathematical Talent based on the Anthropological Theory of Didactics and the notion of mathematical creativity, which defines four technological functions: (1) producing new techniques; (2) optimizing those techniques (3); considering tasks from diverse angles; and (4) adapting techniques. Using this model, this study analyses the creative mathematical activity of students aged 10–12 years displayed as they sought to solve a series of infinite succession tasks proposed to encourage the construction of generalization processes. The setting is a Mathematics Club (a talent-promoting institution). The evaluation of results shows that the Praxeological Model of Mathematical Talent allows the emergence and analysis of mathematical creativity and, therefore, encourages the development of mathematical talent.

**Keywords:** mathematical potential; mathematical talent; mathematical creativity; generalization

## 1. Introduction

For several decades, diverse models and definitions of talent have been proposed in the literature. Singer, Sheffield, Freiman and Brandl point out that most of those models and definitions describe talent as the potential to successfully perform a certain activity, and indicate that while a broad range of models of mathematical talent exist, they are still limited [1]. Research devoted specifically to defining mathematical talent emphasize three areas of study: (1) the characteristics of subjects with talent [2–4]; (2) differences in the characteristics of “talented” vs. “good” students [5,6]; and (3) perspectives on creativity as a fundamental element of talent [7,8].

Generally-speaking, however, mathematical talent has been defined on the basis of the subjects’ abilities. Krutetskii specified that:

Mathematical giftedness is the name we shall give to a unique aggregate of mathematical abilities that opens up the possibility of successful performance in mathematical activity (or with school children in mind, the possibility of a creative mastery of the subject). [9] (p. 77)

This perspective on talent stresses the importance of creative dominion in mathematical activity. In contrast to other postures that, as Leikin points out, associate mathematical talent with the innate characteristics of certain individuals, our work seeks to demonstrate that talent is not a “given” quality, but can be developed under certain circumstances and with adequate accompaniment [10,11]. This leads to the following definition of potential talent:

Talent that has not yet been developed or evidenced; that is, the potential exists to develop and demonstrate it, but due to one or more factors have been unable to manifest it in action schemes. [12] (p. 27, translation by the author)

This definition underscores a transcendental aspect of talent; namely, that “it is developable” [11,13], though it is not clear how this development can be achieved. Studies have revealed certain characteristics strongly-related to talent, such as creativity, which is defined as a person’s capacity to produce new ideas, focuses or actions and then manifest them by moving from thinking to reality [14]. At the professional level, mathematical talent is understood as outstanding performance that leads to discoveries and so is closely related to mathematical creativity [10]. Research has demonstrated that developing creative thinking in mathematics is closely connected to the progress of society [15]. In schools, then, mathematical creativity should be included when developing mathematical talent to prepare students to be innovative and reflexive leaders in diverse fields of knowledge that can promote new methods and atypical perspectives in the evolution of science [1].

Leikin sustains that research on the education of subjects with talent and creativity should be oriented by two interrelated aspects: one theoretical, the other applied [16]. The former seeks to understand the nature of creativity and mathematical talent, while the second seeks to foster mathematical creativity and develop mathematical talent. This requires, on the one hand, confronting open, challenging, novel (non-routine) problems [2,4,9,17–20] and tasks that lead students to problem posing [20], and, on the other, determining the conditions necessary to develop creative thinking, such as forging connections with previously-learned material [10], taking distinct paths to solve the same task [8], or discovering the multiple solutions that a task may admit [21]. The latter entail the development of flexibility, a trait long recognized as an essential component of mathematical talent [2,6,8,9,22–26]. These elements are related to the medium of students’ development and their teachers’ or tutors’ didactic and disciplinary knowledge [17,27].

Two current issues stand out on how to nurture talent: differentiated attention, which entails characterizing and identifying subjects with talent [28,29], and an inclusive focus that develops didactic proposals for heterogeneous groups [13,30–33].

These antecedents lead to the following affirmations: (1) mathematical creativity is closely-related to talent; (2) talent is developable; and (3) theoretical models need to be produced to understand the conditions that will make such development possible. As a result, the research questions that guided our study were:

- What type of activities foster developing creativity and, hence, mathematical talent?
- What theoretical model allows us to best analyse creative mathematical activity?
- What institutional conditions propel the development of creative mathematical activity?

To examine these issues, this work established relations between elements of the Anthropological Theory of Didactics (ATD) and the concept of mathematical creativity that led this study to propose a theoretical approach called the Praxeological Model of Mathematical Talent (PMMT). This instrument is elucidated in the following section.

## 2. Theoretical Framework

This section first presents elements of the ATD, an approach that analyses the institutional dimension of human activity [34]. Second, it clarifies the notion of creativity [8,35–37]. These are the basic elements this study utilized to propose a theoretical model of mathematical talent.

### 2.1. Definitions: Institution, Praxeology, and Levels of Complexity

An institution is defined as “a stable social organization that makes it possible to efficaciously confront problematic tasks thanks to the material and intellectual resources it provides to subjects, and the restrictions defined for executing those tasks” [38] (p. 85, translation by the author). In

this sense, a program of attention to talent (e.g., a science fair or Olympic club) can be considered institutions because they offer subjects conditions and resources to confront problems that are framed as elements of human activity. The ATD analyses this using the four components of praxeology, that is, task type,  $T$ ; technique,  $\tau$ ; technology,  $\theta$ ; and theory,  $\Theta$ . Task type ( $T$ ) refers to “what is done”; technique ( $\tau$ ) is “how it is done”; technologies ( $\theta$ ) are “the discourses that produce, explain, and validate the techniques”; and theory ( $\Theta$ ) refers to “broader discourses that produce, explain, validate, and justify the technologies” [39]. Praxeologies have different levels of complexity. The first, and most elemental, is called “punctual” [ $T, \tau, \theta, \Theta$ ], which is defined as having a task type,  $T$ ; a technique,  $\tau$ ; a technology,  $\theta$ ; and a theory,  $\Theta$ . The second is called “local”, and includes various punctual praxeologies [ $T_i, \tau_i, \theta, \Theta$ ], but the same technology,  $\theta$ ; for example, finding the joint solution of a system of lineal equations using only the Gauss-Jordan technique that corresponds to a punctual praxeology, while considering all available techniques (equalization, elimination, substitution, iterative methods, etc.) to find the joint solution of that system would constitute a local praxeology. In this case, the regional praxeology would be Lineal Algebra, the global praxeology is Algebra per se, and the disciplinary praxeology, the field of mathematics. One postulate of the ATD is that producing mathematical activity entails constructing, or reconstructing, at least one local praxeology [39], so any proposal for developing mathematical talent must include this element of praxeology; that is, a praxeology in which tasks can be performed using techniques produced by the same mathematical technology.

In order to identify the characteristics of creative mathematical praxeologies—that is, those in which creativity plays a specific role—the following section analyses the notion of mathematical creativity.

## 2.2. Mathematical Creativity

In Mathematical Education, the characterization and identification of creativity is often related to four components: fluency, flexibility, originality [8,36,37], and elaboration [35]. ‘Creativity’ is a term widely used to refer to the capacity to produce new ideas, focuses, or actions, and then manifest them by transiting from thought to reality [14]. Sriraman, one of the author/editors of the book, *Creativity and Giftedness*, proposes a significant differentiation between creativity at the professional and school levels [40]. There, professional mathematical creativity is defined as the “(a) the ability to produce original work that significantly extends the body of knowledge, and/or (b) the ability to open avenues of new questions for other mathematicians” (p. 23), while school-level mathematical creativity refers to:

“(a) the process that results in unusual (novel) and/or insightful solution(s) to a given problem or analogous problems, and/or (b) the formulation of new questions and/or possibilities that allow an old problem to be regarded from a new angle requiring imagination”. (p. 24)

Our approach adopts the second conceptualization to define the theoretical model of mathematical talent that is described in detail in the next section.

## 2.3. Praxeological Model of Mathematical Talent (PMMT)

The PMMT considers the aforementioned, classic components of praxeology: task type,  $T$ ; technique,  $\tau$ ; technology,  $\theta$ ; and theory,  $\Theta$ ; but distinguishes two elements of technology, one mathematical, the other creative. This study associates the technological mathematical component,  $\theta^m$ , with what other conceptual frameworks classify as mathematical abilities (reasoning mathematically, economizing thought, expressing logical and sequential thought, abstraction, and generalization, among others); elements associated mainly with mathematical institutions. The creative technological component  $\theta_c$ , in contrast, relates to the production of unique, unusual, flexible or perspicacious techniques that integrate elements from experiences in various institutions, such as family, school, the street, and clubs, among others.

The PMMT is represented as a creative praxeology in which  $P_c = [T, \tau, \theta_c^m, \Theta]$ . In order to analyse the creative mathematical activity, in this model, the technology ( $\theta_c^m$ ) is divided by two components:

The mathematical component ( $\theta^m$ ) corresponds to the mathematical technology from the classic model  $[T, \tau, \theta, \Theta]$  sustained by mathematical theory, while the second ( $\theta_c$ ) reflects creativity understood through the following four technological functions:

*F1. Producing unique techniques:* Faced with a novel task, distinct steps are produced without following a specific routine. Exploring the task presented in relation to that which is already known generates new ideas and leads to constructing new techniques, all guided by the assignment.

*F2. Optimizing the technique:* Evaluating a range of possible routes that allow performance of the task, then choosing the “optimal” one as a function of the number of steps and mathematical knowledge involved. This may mean constructing a general rule (verbal, iconic, or alphanumeric) instead of using a series of drawings to determine the properties of an unknown stage in a figural sequence.

*F3. Considering tasks from diverse angles:* This means analysing the task without restricting it to a certain domain (such as algebra, geometry, discrete mathematics, etc.), or discipline (physics, chemistry, visual arts, etc.), either by producing steps that permit task execution (recognizing the knowledge that motivated the subject to follow a certain path or to change direction), or generating diverse techniques to perform the same task.

*F4. Adapting a technique:* This entails identifying the functioning, scope, and limits of a technique produced in order to implement it on another task after certain modifications. First, the technique must be validated (that is, verifying that its steps make it possible to do what is proposed); second, it needs to be adapted and, perhaps, improved while solving another task.

These functions are closely interrelated during the solving process; for example, the function of producing unique techniques (F1) may emerge when adapting a technique (F4), while considering tasks from diverse angles (F3) may generate unique techniques (F1). Hence, these functions do not follow a fixed order during their development, but are assumed as key elements that permit the analysis of creative mathematical activity.

#### 2.4. Implementing the Praxeological Model $P_c = [T, \tau, \theta_c^m, \Theta]$

This investigation examines a local-level praxeology called “infinite successions”. The task type assigned,  $T$ , is “to determine the  $n$ th stage of an infinite succession”. The technique,  $\tau$ , consists in:

1. Studying the first cases or stages and determining the relation between  $a_i$  and  $i = 1, 2, \dots, n$ .
2. Constructing a general rule for the  $n$ th stage.
3. Implementing the rule constructed in (2) for specific close and distant stages of the succession ( $n = 10, n = 20, n = 100$ ).

The creative technology,  $\theta_c$ , is associated with the four aforementioned technological functions, while the mathematical technology,  $\theta^m$ , is determined by the process of recursive and/or algebraic generalization:

*Recursive generalization* appears when coincidences are observed in the base stages (perceptual field). On this basis, a relation of dependence is considered between  $a_n$  and  $a_{n+1}$ . Later, a rule is constructed for the stage  $a_{n+1}$  based on the stage  $a_n$  (inferential field). This makes it possible to determine “close” values [41–44].

*Algebraic generalization* appears when a verbal rule or expression is constructed on the basis of the stages of the perceptual field or recursive rule that determines a relation of dependence between any stage  $n$  and the stage of the sequence  $a_n$  (with no need to obtain the previous term,  $a_{n-1}$ ). This permits the determination of both “close” and “distant” values [41–44].

Generalization is taken as the mathematical technology,  $\theta^m$ , because it is determinant in constructing a general rule for a succession, be it figural, numeric, or figural with a tabular aid. In other praxeologies, technology may be associated with logical-sequential thought or abstraction.

Finally, the theory,  $\Theta$ , in this model is defined by Combinatory Theory.

### 3. The Study

#### 3.1. The Setting and Characteristics of the Institution

This is a qualitative investigation that adopted a case study methodology to analyse creative mathematical activity at an institution that promotes mathematical talent. The institution is called the Math Club (MC), and it was formed with the aim of implementing a series of tasks of the same type—local praxeology—that together enable creative mathematical activity.

*Institution, the MC.* The MC was conceived and constituted by one of the authors of this article with the support of the government of Baja California (México) during the development of her doctoral thesis. The participants were children from 15 public schools in the state, their parents, and an expert in mathematical didactics. The MC held 10 weekly 90-min sessions that were videotaped and transcribed for a deeper analysis of the creative mathematical activity that was potentiated using the PMMT proposed herein.

*Students.* The students were selected by applying a written test that presented four math problems. In addition to the results of the test, the authors took into account the recommendations of math teachers regarding the interest of different students in participating in the MC. Table 1 summarizes the characteristics of the group.

**Table 1.** Characteristics of the participants in the Math Club.

	5th Grade	6th Grade		
Sex	10 Years	11 Years	12 Years	Total
Female	2	8	2	12
Male	3	8	3	14
Total	5	16	5	26

*The role of the instructor.* The instructor's participation consisted of constructing the didactic proposal presented below, after considering previously-analysed didactic variables that were transformed according to the students' needs. Throughout the study, the instructor emphasized that all ideas, first explorations, and embryos of techniques were important, and motivated the students to explore cases, explain their techniques, and then determine their validity. The instructor monitored the students' work, guided group meetings, and promoted the collective validation or refutation of the different techniques proposed. The instructor's participation was indispensable in establishing a climate of respect and collective analysis.

*The role of parents.* Though the parents only participated actively in the first (presentation of the MC) and last sessions (when parents and children played a mathematical game called "double domino"), their role in accompanying their children and ensuring their regular participation in the MC was fundamental to the success of the study.

The inputs obtained as the basis for analysing the participants' creative mathematical activity consisted of the following: (1) their worksheets; and (2) transcriptions of selected "episodes" from the videotaped sessions. Episodes are defined as "work intervals" in which pairs of students sought to construct or present the mathematical and creative technological elements that were analysed using the PMMT, as shown in the Results section.

#### 3.2. Didactic Design of the MC

The personal characteristics of a subject such as good memory, commitment to the task, and a positive attitude towards mathematics, all increase the possibilities of enhancing mathematical talent. One way to ensure that these characteristics participate in developing such talent is to have students confront a didactic design that consists of a set of challenging, interrelated mathematics tasks in an institutional setting that propitiates the development of creativity. The present study proposed

a didactic design based on an “infinite successions” praxeology with six challenging problem situations adapted from earlier studies of mathematical talent. Table 2 briefly describes the tasks included in the six situations.

**Table 2.** Tasks presented in all the MC sessions.

Problem Situation	Tasks
1. Sierpiński’s triangle	Task 1.1: Determine the change in the number of triangles while building Sierpiński’s triangle. Task 1.2: Determine changes in the area while building Sierpiński’s triangle. Task 1.3: Determine changes of the perimeter while building Sierpiński’s triangle.
2. Origami cubes	Task 2.1: Determine the number of cubes required to construct any stage of the sequence (different sequences of origami cubes were presented). Task 2.2: Construct three stages using the origami cubes following a pattern.
3. Tables and chairs	Task 3.1: Determine the number of chairs that can be arranged at any number of tables. Task 3.2: Determine the number of squares required to build any stage of the sequence. Task 3.3: Determine the number of shaded squares required to construct any image.
4. Intersections and regions	Task 4.1: Determine the number of intersections formed on a plane when any number of non-parallel lines are placed but only two lines can form an intersection. Task 4.2: Determine the number of regions formed on a plane when any number of non-parallel lines are placed but only two lines can form an intersection.
5. Tower of Hanoi	Task 5.1: Determine the minimum number of movements with any number of disks.
6. Folding paper	Task 6.1: Determine the total thickness of the folded sheet for any number of folds. Task 6.2: Determine the surface area of the sheet for any number of folds. Task 6.3: Determine the perimeter of the surface of the sheet for any number of folds.

The tasks defined for each situation (Table 2) permitted constructing lineal, quadratic, or exponential general rules. Task 6.3 has the most challenging algebraic generalization rule:  $a_n = 32/(2^{n-1})$  for all  $n \in \mathbb{N}$ . It is important to clarify that this study did not expect the students to construct expressions with alphanumeric algebraic syntaxes, but generalizations of algebraic type, understood from the perspective of Radford as factual, contextual, or symbolic [40]. Though all six situations shared the goal of reaching an algebraic expression, certain specific elements were not common: for example, the use of didactic material (situations 2, 4, 5), problem posing (situation 2), and managing concepts foreign to traditional instruction (situation 1).

These problem situations were organized with the expectation that they would foster the emergence of a creative component that could be analysed using the four technological functions outlined above. As elucidated below, each situation was accompanied by several tasks designed to generate a mathematical activity. While the analysis of the mathematical activity generated by a specific task may not necessarily discern all four creative functions, together these tasks fomented creative mathematical activity and, hence, the development of mathematical talent.

The next section focuses on situations 2 and 4, which were used in this study to evidence the creative component of the students’ activity and how it can be analysed using the PMMT.



#### 4. Results

The results presented in this section reflect our analysis of the mathematical activity of three pairs of students. This study chose situations 2, “Origami Cubes”, and 4, “Intersections and Regions”, which were developed in sessions 2 and 4–5, respectively, because they allowed the authors to analyse the evolution of creative mathematical activity using the PMMT model.

##### 4.1. Problem Situation 2: Origami Cubes

This problem included two tasks:

- Task 2.1: Determine the number of cubes required to construct any stage of the sequence (different sequences of origami cubes were presented).
- Task 2.2: Construct three stages using the origami cubes following a pattern.

Task 2.1 consisted in determining, based on the first three stages of a sequence, a general rule that would define all the stages of the sequence. Task 2.2 involved building, with tangible materials (origami cubes), three stages of any sequence. To perform Task 2.1, the students were shown four different construction sequences of the cubes, some with lineal generalization rules, others with quadratic rules. One of the quadratic rules is shown below (Figure 1).

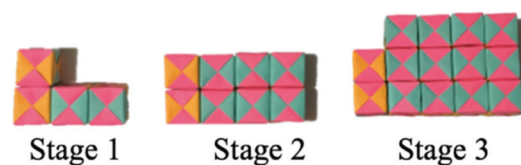


Figure 1. Construction of origami cubes presented to the students of the MC.

The technique required to solve this task entails constructing a relation between the number of cubes in each figure generated for each term, following three main steps: (1) studying the first three stages to construct an initial rule that determines a relation between the number of cubes and the stage of the sequence (Figure 2); (2) constructing a rule for the  $n$ th stage (Table 3); and (3) verifying the general rule for the stage  $n + 1$ , which can be demonstrated using the induction principle and, for these students, proving that rule for different stages.

Table 3. Arrangements of the cubes required to construct the  $n$ th stage of situation 2.

Term	Combination Pink/Orange	Combination Pink/Blue	Total Cubes	Increase of Cubes between Stages
1	2	2	4	
2	2	6	8	+4
3	2	12	14	+6
4	2	20	22	+8
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$n$	2	$n(n + 1)$	$n(n + 1) + 2$	$+2n$

The cubes and the table containing the combinations of colours and totals (Figure 2) constitute two ostensives, that is, palpable records according to Bosch and Chevallard [45]. They allow the development of the technique and technology. Specifically, the table (Figure 2) allowed students to “decompose” [43] (p. 66) the task down into sub-tasks and so control the technique.

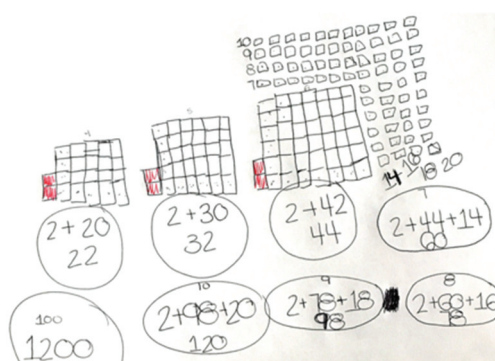
First combination of colors						
Stage 1	Stage 2	Stage 3	Stage 4	Stage 5	Stage 6	Stage 100
Second combination of colors						
Stage 1	Stage 2	Stage 3	Stage 4	Stage 5	Stage 6	Stage 100
Total						
Stage 1	Stage 2	Stage 3	Stage 4	Stage 5	Stage 6	Stage 100

**Figure 2.** Table to determine the number of cubes according to the combinations of colours.

Tables 4–6 show the techniques and technologies produced by the three pairs of students that were representative and contrasting in terms of the mathematical and creative technologies in play.

**Table 4.** Pair A's techniques and technologies related to Task 2.1.

Technique	Technologies $\theta^m$ and $\theta_C$
<p>Step 1. Determine that the first combination (orange/pink) remains unchanged, and place the number 2 in the first combination up to stage 100 (Figure 2).</p> <p>Step 2. Count the cubes in the second combination (blue/pink) and write the numbers 2, 6 and 12, respectively, in the first three cells.</p> <p>Step 3. Identify a pattern based on the stages given and make drawings with squares to represent the cubes for stages 4, 5 and 6 in the two combinations (Figure 3).</p> <p>Step 4: Based on the drawing of stage 6, they made drawings of the increments for <math>n = 7</math>, <math>n = 8</math>, <math>n = 9</math> and <math>n = 10</math> (Figure 3). Next they counted and made the sums.</p> <p>Step 5: To obtain the result for stage 100, they multiplied the number of cubes of stage 10 by 10. The instructor asked them to prove this 'rule of 3' for stage 9 based on stage 3. After doing so, they abandoned this technique. Later, they returned to their attempt to count the number of cubes using drawings.</p>	<p><math>\theta^1</math> : On the first three stages, they explored other close ones using drawings, but could not determine a recursive rule <i>per se</i>.</p> <p><math>\theta_1</math> : They optimized the technique by changing from drawing the complete figure to drawings that show only the increments (F2).</p> <p><math>\theta_2</math> : They tried to optimize (F2) by adapting a school technique called 'the rule of 3' to determine case 100, but failed to solve the task.</p>



**Figure 3.** Pair A's drawings to solve Task 2.1.

Table 4 shows a first approximation towards the creative mathematical activity defined by F2 and F4. To determine the stage  $n = 100$ , pair A optimized the drawing technique of each figure by considering only the increments. They then sought to optimize utilizing the school technique called the rule of 3. Their return to the tangible material led the authors to suppose that they tried to obey the rules that govern the institution of the CM. Their adherence to the physical material of the cubes and drawings did not allow them to reach a rule of recursive generalization. The development of creativity is associated with the task type and the implicit rules established to execute it, as shown below.



In what follows (Table 5), the study presents the technique used by pair B and its associated technologies.

**Table 5.** Pair B's techniques and technologies related to Task 2.1.

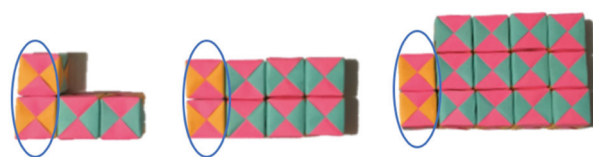
Technique	Technologies $\theta^m$ and $\theta_C$
Step 1. Same as for pair A. Step 2. Same as for pair A. Step 3. Based on the initial stages of the second combination, they determined that stage 1 has 2 cubes, stage 2 has $2 + 4 = 6$ cubes, and stage 3 has $2 + 4 + 6 = 12$ cubes. They then departed from the material and made a sum to determine the number of cubes in stage 4: $2 + 4 + 6 + 8 = 20$ cubes. Step 4. Based on this algebraic rule, they attempted to implement this for $n = 100$ , proposing the operation: $2 + 4 + 6 + 8 + \dots + 200$ ; however, they were unsuccessful in calculating the operation or optimizing their technique.	$\theta^1$ : Based on the initial stages, they constructed an algebraic rule for distant stages as follows: $a_n = 2 + 4 + 6 + \dots + 2n$ . $\theta_1$ : They optimized the technique (F2) by setting the cubes aside and focusing on the sum required to determine the number of cubes in the second combination.

The creative mathematical activity in this pair is manifested in their optimization of the technique (F2). These students set the tangible material aside and used the data in the Table to construct a technique and then implement it for the stages  $n = 4$  and  $n = 100$ . Though this technique is uneconomical due to the arithmetic calculations required to reach the result, it did solve the task.

Table 6 shows pair C's technique and associated technologies.

**Table 6.** Pair C's techniques and technologies related to Task 2.1.

Technique	Technologies $\theta^m$ and $\theta_C$
Step 1. Same as for pair A. Step 2. They related the origami cubes in combination two to the floors of a building, observing that 'In stage 1 there are 2 columns of 1 floor, in stage 2 there are 3 columns of 2 floors, in stage 3 there are 4 columns of 3 floors'. Step 3. They determined that stage 4 of combination two has 5 columns of 4 floors. Using this recursive rule, they completed the row of combination two up to stage 100: 101 columns by 100 floors: $101 \times 100 = 10,100$ . Step 4. They noted that in the table (Figure 2) only 2 cubes must be added to the total number of cubes in combination two (Figure 4). They then calculated the sum that corresponds to the total up to stage 100, which is $10,100 + 2$ . Step 5. Upon the instructor's indication, they determined another technique to solve the task by identifying that 'squares' are formed in combination two and calculating their respective area; that is, for the first stage, a $1 \times 1$ square, for the second, a $2 \times 2$ square, and for the third, a $3 \times 3$ square. They then proposed that to each calculation of the product they needed to add 1 to the first term, 2 to the second, and 3 to the third (Figure 5). They concluded that: 'the number of the stage is multiplied twice, the new stage is added, and then 2 from the other combination is added'.	$\theta^1$ : Based on the initial stages, they determined an algebraic rule with two equivalent patterns of construction: $a_n = n(n + 1) + 2$ $b_n = n \times n + n + 2$ . $\theta_1$ : To optimize the technique, they generated a novel technique using a known reference; namely, a building to represent the cubes (F1, F2, F4). $\theta_2$ : They produced two different techniques to perform the same task (F3).



**Figure 4.** Pair C's reconfiguration of the sequence in Figure 1.

Pair C manifested creative mathematical activity by optimizing a technique based on counting and generating a novel technique; that consisted of determining the number of cubes by multiplying the number of the stage by the number of the stage +1. This entailed relating and adapting the rows of cubes to the floors of a building. This relation did not emerge from just their knowledge of math, but involved aspects of real life (F1, F2, F4). These students adapted the decomposition technique presented in Task 2.1 (based on identifying the stages of the sequence by separating the two colour combinations) to deconstruct [43] each stage into three distinct organizations (Figure 5): a constant (2), a square ( $n \times n$ ), and a line ( $n$ ). This enabled them to propose a new technique to solve the task that evidences the development of F1, F3 and F4.

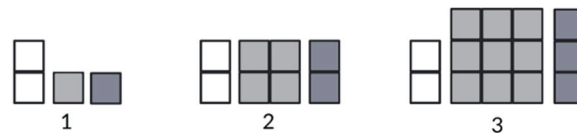


Figure 5. Pair C's second reconfiguration of the sequence of Figure 1.

When pair C finished, the instructor asked them to explain their techniques to the group, and then continue with Task 2.2, which involved constructing three figures of a sequence with the origami cubes following a pattern similar to the previous ones. They were informed that other teams would have to determine the number of cubes required to construct stage 100 of the sequences posited. Figure 6 shows some of the sequences they constructed.

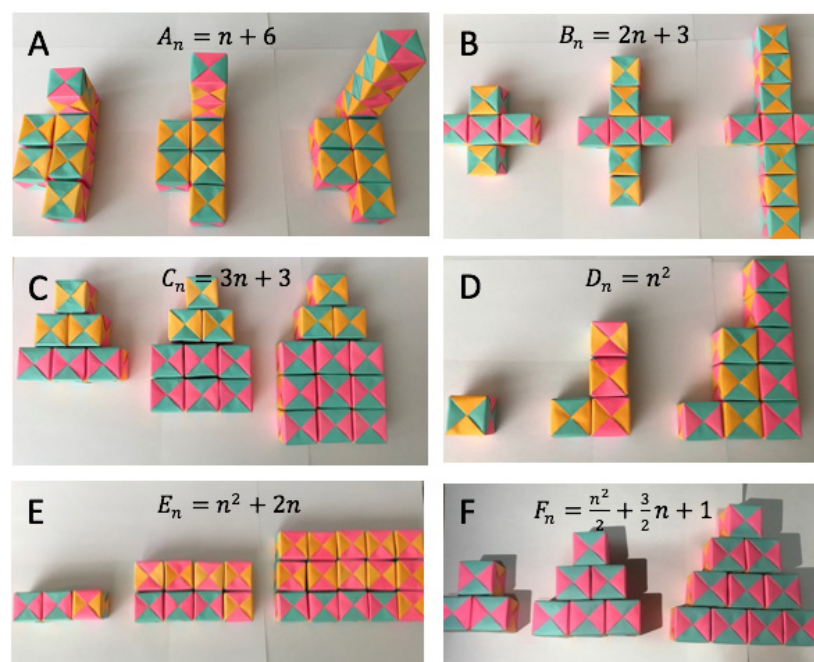


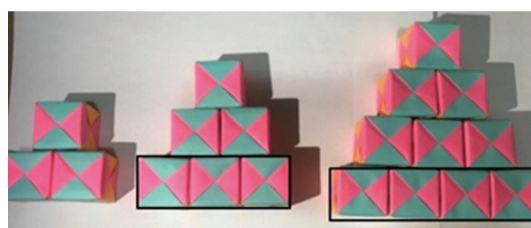
Figure 6. Construction with cubes by the MC students for Task 2.2.

Creative mathematical activity was observed when all the students proposed tasks of the same type; that is, tasks that can be solved using techniques associated with the same technology. The generalization rules that determine the  $n$ th stage of the constructions of cubes created by the students were of two types: lineal and quadratic. For the first five sequences constructed, the pairs of students found a rule of algebraic generalization associated with the  $n$ th term. Sequence F, which they called “pyramids” (Figure 6), was presented at a first moment to one pair of students, but they were unable to determine the rule of algebraic generalization, so the instructor presented the rule to all the students at the end of the session so they were able to solve the task as a group. This task was:

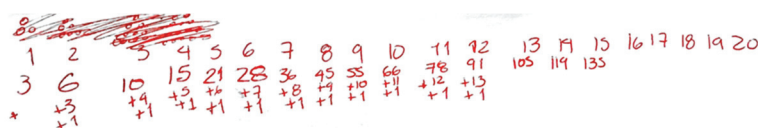
how many cubes are needed to construct stage 100 of the sequence of pyramids? Table 7 shows the techniques and technologies that emerged from the group mathematical activity.

**Table 7.** Group techniques and technologies related to Task 2.2.

Technique	Technologies $\theta^m$ and $\theta_C$
<p>Step 1. By manipulating the material, they determined that a row of 3 cubes was added between the stage 1 and stage 2 pyramids, and that a row of 4 cubes was added between the stage 2 and stage 3 pyramids (Figure 7).</p> <p>Step 2. Next, they determined that a row of five cubes had to be added below pyramid 4 and, similarly, a row of six cubes had to be added below pyramid 5.</p> <p>Step 3. They determined that calculating the number of cubes of any pyramid of this sequence required totalling the number of cubes of the previous pyramid and the stage + 1 (Figure 8). Using this rule, they constructed a list up to stage <math>n = 15</math>.</p>	<p><math>\theta^1</math> : Based on the initial stages, they determined the following recursive generalization:  <math>a_{n+1} = a_n + (n + 1)</math>.</p> <p><math>\theta_1</math>: They optimized the technique of counting into a recursive technique (F2).</p>



**Figure 7.** Representation of the technique for the pyramid task.



**Figure 8.** List of the first 15 stages for Task 2.2.

The students' creativity in this activity was observed when they optimized the counting technique (F2). Beyond that, their posing of these constructions evidenced embryos of creative technologies associated with their mathematical activity; namely, recognizing the characteristics of tasks of the same type with their associated technology. Though no pair of students determined a rule of algebraic generalization that corresponds to the stages of the sequence where the number of cubes of the  $n$ th stage of the sequence of pyramids is defined by  $a_n = \frac{1}{2}n^2 + \frac{3}{2}n + 1$ , it was suggested to keep thinking about this task after the session. Member 2 of pair C constructed a table that included all the stages from  $n = 1$  to  $n = 100$  based on the recursive rule,  $a_{n+1} = a_n + (n + 1)$ , which turned out to be very useful for performing Task 4.1, as shown below. In general, the students' experience with, and reflections on, constructing the techniques for tasks 2.1 and 2.2 became a creative technological element that allows the development of other techniques for this task and their adaptation to perform other types of tasks.

#### 4.2. Problem Situation 4: Intersections and Regions

This situation, or a variant of it, has been used in other studies, especially programs of attention to talent (for example, in ESTALMAT; see <http://www.estalmat.org>).

- Determine the number of regions into which a plane is divided by  $n$  non-parallel lines, where an intersection can only be formed by two lines.

The technique required to solve this task entailed identifying the regions generated by the intersections of a family of  $n$  non-parallel lines, where there is no common point in three different lines. The tasks assigned involved determining the number of intersections and regions. To this end, the students were initially given six pieces of pasteboard as in Figure 9.

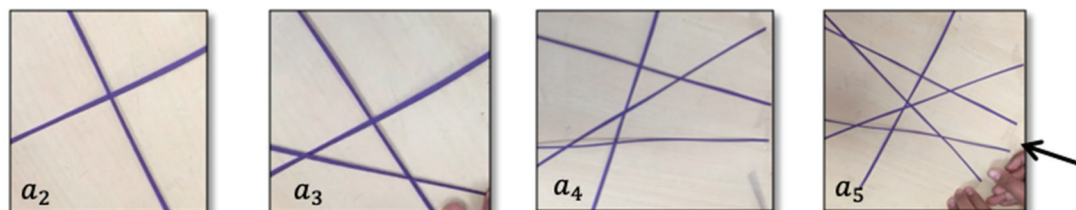


Figure 9. Evidence of the ostensive delivered to the students for situation 4.

The arrow in Figure 9 at  $a_5$  signals an intersection that is not visible, but that would appear if the two-line segments were prolonged. While studying these cases, then, it was important to consider the infinity property of lines on a plane so as to correctly count all the intersections and regions. In addition to the tangible material, the instructor posed several questions designed to mobilize the technological elements associated with the two tasks; that is, parallelism and non-parallelism between lines, the infinity property of the lines and plane, the intersection of two lines, and the regions formed by those intersections. Both the questions and the material functioned to remind the students to consider, as well, the intersections that did not appear visually on the plane by prolonging the lines to generate the first cases. In the first step of the technique, a rule was constructed to determine the relation between the number of lines and regions generated. Figure 10 represents the first five cases.

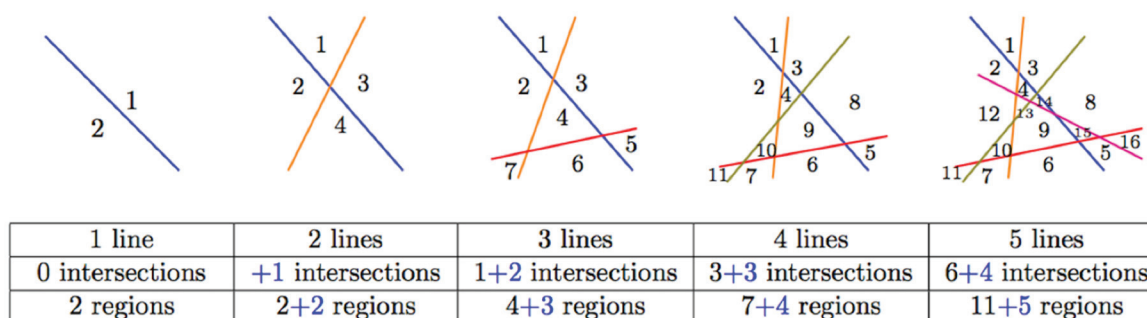


Figure 10. Construction of the five first cases of the intersections task.

In a second step, studying the  $n$ th case, students constructed the relation between the number of lines and regions generated when  $n$  lines were placed in accordance with the task conditions (Table 8). The third step consisted in verifying that the rule posited in step two functioned for  $n + 1$  lines. In this case, the students were able to prove that the rule functioned for any number of lines ( $n = 5, n = 20, n = 100$ ).

Table 8. Construction of regions on the plane.

Lines	$a(n)$	Increase of Regions
1	2	
2	4	+2
3	7	+3
4	11	+4
$\vdots$	$\vdots$	$\vdots$
$n$	$1 + \frac{n(n+1)}{2}$	$+n$

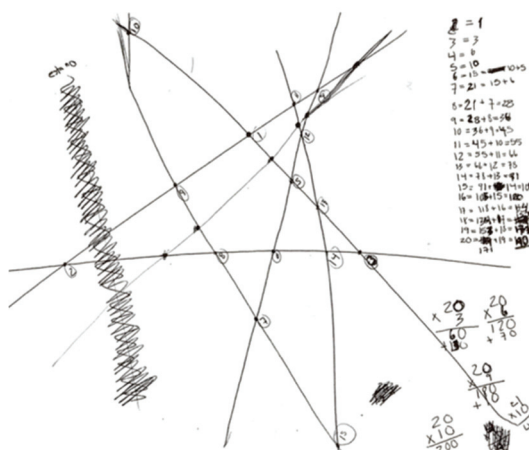
This problem situation included the following two tasks:

- Task 4.1: Determine the number of intersections formed on the plane when any number of non-parallel lines are placed and only two lines can form an intersection.
- Task 4.2: Determine the number of regions formed on the plane when any number of non-parallel lines are placed and only two lines can form an intersection.

As the session progressed, but before these tasks were presented, the instructor posed a series of questions that included: How many regions are formed if 100 lines are placed but with only one intersection? The aim of these questions was to mobilize the technological elements associated with the two tasks; that is, parallelism and non-parallelism between lines, the infinity property of the line and plane, the number of intersections between lines, and the regions formed by those intersections. Here, this study discusses the techniques and technologies produced by pair A in relation to Task 4.1 (Table 9).

**Table 9.** Pair A's techniques and technologies related to Task 4.1.

Technique	Technologies $\theta^M$ and $\theta_C$
<p>Step 1. They used the tangible material to count the intersections for the first 6 lines, but failed to consider the non-visible intersections. The instructor pointed out this error, which led to step 2.</p> <p>Step 2. They made drawings of the lines on a blank sheet (Figure 11) to count all the intersections more systematically. This allowed them to count all the intersections formed by drawing seven lines.</p> <p>Step 3. They constructed a recursive process for the first 8 lines: 6 lines generate 15 intersections (<math>10 + 5</math>); 7 lines produce 21 lines (<math>15 + 6</math>). Hence, 8 lines should give <math>21 + 7 = 28</math> intersections. They verified the number of intersections by following the conditions on the list (Figure 11) and then used this technique to reach 20 lines.</p> <p>Step 4. They made several attempts to find a rule based on the relation between the number of lines and the number of intersections of the stage <math>n = 20</math>, and tested their proposed rules with the information for stage 4, but did not succeed in establishing an algebraic rule.</p>	<p><math>\theta^1</math> : Taking the first 6 stages as the base, they considered a relation of dependence between <math>a_n</math> and <math>a_{n+1}</math>, and, later, the recursive rule:  <math>a_n = a_{n-1} + (n - 1)</math>.</p> <p><math>\theta_1</math> : They optimized their technique by passing from the drawings to constructing a recursive rule (F2).</p> <p><math>\theta_2</math> : They attempted to optimize their recursive technique and find a rule of algebraic generalization (F2). They also applied the verification technique presented previously (F4).</p>



**Figure 11.** Graphic representation of pair A's technique for the intersections task.

In contrast to the previous task, this case requires a construction based on a process of exploring the first stages that define the relation between the number of intersections ( $a_i$ ) and lines ( $i$ ). For this reason, the students changed the technique of using the material to one based on “drawing lines”.



This allowed them to count all the intersections (F3). Creative mathematical activity also appeared when they proposed a recursive process to determine the number of lines, thus optimizing their technique (F1, F2). Upon contrasting step 5 of Task 2.1 with step 4 of this one, this study observed that the students adapted the technique (F4) when they proposed diverse rules of algebraic generalization, and when they verified those rules for close and distant stages in the sequence.

Table 10 displays the creative mathematical activity based on pair B's techniques and technologies with Task 4.1.

**Table 10.** Pair B's techniques and technologies related to Task 4.1.

Technique	Technologies $\theta^m$ and $\theta_c$
<p>Step 1. Using the tangible material, they counted the number of intersections for the first 5 stages, including the intersections that do not appear visually (Figure 5).</p> <p>Step 2. They constructed a recursive process for the first 8 lines based on the differences between stages <math>a_n</math> and <math>a_{n+1}</math>; that is: with 2 lines, one intersection; with 3 lines, <math>1 + 2 = 3</math> intersections; with 4 lines, <math>3 + 3 = 6</math> intersections; with 5 lines, <math>6 + 4 = 10</math> intersections; and with 6 lines <math>10 + 5 = 15</math> intersections; so with 7 lines the number of intersections will be the same as with 6 lines plus 6: <math>15 + 6 = 21</math> intersections, and with 8 lines the number will be the same as with 7 lines plus 7: <math>21 + 7 = 28</math> intersections (Figure 12).</p> <p>Step 3. They identified that the technique associated with the pyramid task (Figure 4) can be applied to solve the intersection task, as well, though they recalled that the earlier task was not solved.</p> <p>Step 4. They determined which task is solved by adding up consecutive positive whole numbers, such that <math>a_{100} = 1 + 2 + 3 + \dots + 99</math>.</p> <p>Step 5. They related a famous technique from Carl Friedrich Gauss (Hayes, 2006) to add up the first consecutive positive numbers (Figure 13) that the instructor had mentioned briefly in the first session.</p> <p>Step 6. They determined the value of the sum <math>1 + 2 + 3 + \dots + 99</math>, by forming pairs between the 10 initial and 10 final numbers, as shown here:  <math>1 + 99 = 100</math>; <math>2 + 98 = 100</math>; <math>3 + 97 = 100</math>; <math>4 + 96 = 100</math>; <math>5 + 95 = 100</math>; <math>6 + 94 = 100</math>; <math>7 + 93 = 100</math>; <math>8 + 92 = 100</math>; <math>9 + 91 = 100</math>; <math>10 + 90 = 100</math>; until they obtained <math>100 \times 10 = 1000</math>.          Finally, they stated that this process must be repeated up to the number 50, but that this technique is not effective for any number of lines stipulated, so they continued trying to come up with an algebraic rule.</p> <p>Step 7. The instructor asked them to perform the same activity, but with 20 lines. This led to the following conclusion: taking away one (leaving 19), dividing by two (leaving 9.5), taking only the 9 (the whole part) and multiplying <math>9 \times 20</math> (making 180) plus the one in the half, leaves 190 intersections.</p> <p>Step 8: They verified this technique with one member's results for 20 lines and then implemented it for 100 lines: <math>99/2 = 49.5</math>, taking 49, multiplying it by 100, and adding 50 from the half to obtain the result of 4,950 intersections.</p>	<p><math>\theta^1</math> : Based on the first 5 cases, they considered a relation of dependence between <math>a_n</math> and <math>a_{n+1}</math> that led to the recursive rule:  <math>a_n = a_{n-1} + (n - 1)</math>.</p> <p><math>\theta^2</math> : They constructed two rules of algebraic generalization for <math>n</math> lines:  <math>a_n = 1 + 2 + \dots + (n - 1)</math>  <math>a_n = [(n - 1)/2] \times (n) + \frac{1}{2}n</math>.</p> <p><math>\theta_1</math> : They optimized their technique by passing from the tangible material to a recursive rule.</p> <p><math>\theta_2</math> : They recognized the applicability of a technique based on considering three types of tasks: the intersection task, the pyramid task, and the task of the sum of the first 100 positive whole numbers (F3, F4).</p> <p><math>\theta_3</math> : They adapted Gauss' technique for the first 10 whole positive numbers for the sum of the first 99 numbers (F4).</p> <p><math>\theta_4</math> : They optimized the technique of adding pairs and constructed a rule of algebraic generalization (F2).</p>

2	= 1
3	= 3
4	= 6
5	= 10
6	= 15
7	= 21
8	= 28

**Figure 12.** Record of pair B's technique



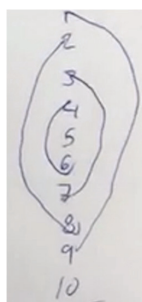


Figure 13. Pair B's adaptation of Gauss' technique.

The development of creativity in this task became apparent when the students identified that the three different tasks, approached in three sessions, could be solved using the same technique (F3, F4). Unlike the case of the technique applied on the previous task (Table 5), on this one, they produced three types of generalizations—one recursive, two algebraic—that solved it (F1, F3), then chose the “optimal” one as a function of the arithmetical calculations required to solve it (F2).

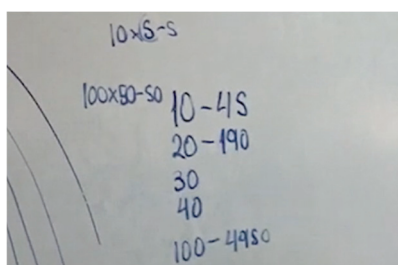
Table 11 presents pair C's creative mathematical activity on Task 4.1.

Table 11. Pair C's techniques and technologies related to Task 4.1.

Technique	Technologies $\theta^m$ and $\theta_C$
<p>Step 1. Using the tangible material, they counted the number of intersections for the first 6 lines, after arranging them so that all intersections were visible.</p> <p>Step 2. Same as pair B, but followed this rule up to 18 lines. Based on this step, they proposed their own techniques, which were discussed at a couple different moments.</p> <p>Step 3. (Member 1). With the goal of detecting a pattern that would allow her/him to construct an algebraic rule, he continued to apply this recursive rule up to 55 lines (Figure 14).</p> <p>Step 3. (Member 2). This student related the technique associated with the pyramid cube task (Figure 4), realizing that it could be used to solve the intersection task. This student showed the instructor a sheet where –on his own initiative– he had resolved the pyramid cube task using the recursive rule <math>a_{n+1} = a_n + (n + 1)</math> one-by-one up to <math>n = 100</math>. He mentioned that the numbers are the same, but ‘go two behind’; that is, ‘If the number of lines here is five, there it would be pyramid 3’. Though this student recognized that this new task is solved using pyramid 98, he continued to search for a more direct way.</p> <p>Step 4. (Member 1). Once this student had a sufficient number of cases (55 lines), he identified a pattern based on ‘tens’ (10, 20, 30, 40 ...). He explained the process on the blackboard (Figure 15): ‘The way in which it is possible to reach 45 is by multiplying <math>10 \times 5</math>—the half— and subtracting 5 to reach 45. I made it to 50 this way, one-by-one and verified it [algebraic rule] for 20, 30 and 40, and it worked; for example, for 20 I obtained 190, because it's <math>20 \times 10</math>, which is half of 20 minus 10, and that gives 190. I did the same with 30, 40 ... up to 100. What I did was <math>100 \times 50</math> minus 50, which is half. That gave me 4950’.</p>	<p><math>\theta^1</math> : Based on the first 6 cases, they structured a relation of dependence between <math>a_n</math> and <math>a_{n+1}</math>, and later constructed the recursive rule:  <math>a_n = a_{n-1} + (n - 1)</math>.</p> <p><math>\theta^2</math> : They constructed a rule of algebraic generalization to find the number of intersections of <math>n</math> lines:  <math>a_n = n\left(\frac{n}{2}\right) - \frac{n}{2}</math>.</p> <p><math>\theta_1</math> : They optimized their technique by passing from the tangible material to a recursive rule (F2).</p> <p><math>\theta_2</math> : Member 2 recognized the applicability of a technique with respect to two task types: pyramid cubes and line intersections. He identified the similarities and differences between the tasks in order to implement the technique (F3, F4).</p> <p><math>\theta_3</math> : Member 1 optimized his recursive technique and produced a unique technique based on ‘tens’. He verified that this new technique worked for other cases (F1, F2).</p>

	28-878	40-780
	29-406	41-820
	30-485	42-861
	31-465	43-903
	32-496	44-946
	33-528	45-990
	34-561	46-1036
	35-595	47-1082
	36-630	48-1129
	37-666	49-1177
	38-703	50-1226
	39-741	51-1266
	40-780	52-1318
	41-820	53-1370
		54-1423
		55-1477
		56-
		57-

**Figure 14.** Record of the recursive rule for the first 55 stages by member 1 of pair C.



**Figure 15.** Written record of the algebraic rule by member 1 of pair C.

As with pair B, member 2 of pair C evidenced his creative mathematical activity by identifying the similarities between the pyramid task with and the intersection task. In addition to identifying these regularities, he recognized the differences in the initial stage of each task and how they would need to be implemented in the new one (F3, F4). The creative mathematical activity was clear in member 1 of pair C when he proposed a peripheral task that economized his technique. Based on the stages in the table (Figure 14), this student focused on multiples of 10 to relate the number of lines with the number of intersections (Figure 14). This evidenced functions F1, F2, F3 and F4.

The series of tasks proposed in this investigation is well-known and has been utilized in early (even pre-algebra) algebra; that is, a sector of the field that is addressed during the transition from arithmetic to Algebra per se. This has often been related to the generalization process [41–44].

## 5. Discussion

Most of the studies or models that set out to analyse mathematical talent and its association with creativity [13,46] tend to centre their gaze on mathematical abilities, types of problems, topics in math, or the mathematical theory that stimulates students' potential. Our PMMT model, in contrast, takes all these elements into account (task type, techniques produced by students, and the mathematical,  $\theta^m$ , and creative,  $\theta_C$ , technologies that come into play), together with their interrelations during the creative mathematical activity. It then characterizes this activity on the basis of the creative technological functions that emerge as protagonists during the analysis of results.

The infinite successions tasks presented herein facilitated the production of diverse techniques for finding recursive generalization and algebraic rules while performing the same task (F3). This approach can thus be compared to flexible thinking (number of focuses of a solution) in the sense elucidated by Kattou, Kontoyianni, Pitta-Pantazi and Christou [8]. The case of pair C revealed creative technologies from the first session (F1, F2, F3, F4), but it was the global analysis that generated evidence of the development of their creativity and of how the other functions became clearer. However, confronting one sole task, as in the context of this investigation, does not allow the authors to “characterize the flexibility” of all the students, as illustrated by the creative mathematical activity reported for pairs A and B, who began developing creativity during the second task, but did not provide clear indications of this until Task 4.1. Thus, presenting a series of tasks of the same type (not identical, but ones that share the same technology), is what made it possible to clearly evidence the systematic development of

creativity and, in this study specifically, the way in which most of the students constructed recursive generalization and algebraic rules by recognizing, modifying, adapting, and applying the same technique to several different tasks. In this sense, proposing a local praxeology based on various tasks of the same type associated with the same technology allowed those techniques to be optimized or adapted to other conditions; for example, the techniques that pairs B and C produced on the pyramid task were implemented to generate the technique for Task 4.1 of the intersections task.

The conditions of proposing a local praxeology—tasks of the same type—aligns with the condition of proposing “challenging” tasks, as has been defined in other research on mathematical talent and creativity [4,14,17–19]. In particular, our work revealed that these MC students developed mathematical creativity through the implementation of a didactic design that emphasized task type, *T*: “determine the *n*th stage of an infinite succession” and, on that basis, developed talent by means of a mathematical activity based on generalization.

In addition, this study observed that certain steps of the technique produced by the students were key to developing mathematical technologies. The study of the first three stages (step one of the technique), for example, made it possible to generate a technological element; search for the pattern associated with the general rule (stage two of the technique). This was evidenced specifically by pair C’s activity, as they recognized that determining any stage of the sequence using a recursive rule is an uneconomical technique, though it allowed them to find a pattern and an algebraic rule that generated any stage of the sequence. In particular, member 1 of pair C showed an embryo of a mathematical technology that led him to elaborate a list with a considerable number of stages that was not limited by a dependency between  $a_n$  and  $a_{n+1}$  but, rather, by choosing the specific stages ( $a_{10}$ ,  $a_{20}$ ,  $a_{30}$ , ...) that were determinant in optimizing the technique. This reveals a general vision of recurrence that, for this student, seemed to occupy a “first level” of generalization; one that he then applied to construct a rule of algebraic generalization to solve the task. A similar situation occurred with member 2 of pair C, who constructed a table containing a set of a data that, once analysed, allowed him to solve the pyramid task for the first 100 stages. This student established a direct relation between the pyramid and intersections tasks, pointing out that: “it goes two behind”. Though this student found  $a_{100}$  and recognized that this new task can be solved with  $a_{98}$ , the technique was guided by the objective of the task; that is, determine any stage of the sequence. Hence, the implicit non-dependence task with the previous stage ( $a_n \rightarrow a_{n+1}$ ) on Task 2.1 was very clear for these two students.

One advantage of using this model, together with generalization as the mathematical component ( $\theta^m$ ), is the flexibility it allows regarding the objectives of each task; that is, the students have the possibility to propose preliminary conclusions for tasks—e.g., “obtaining a recursive rule”—although they do not succeed in constructing an algebraic rule. Moreover, this model allows them to produce diverse techniques for the same task (F3), either on their own initiative or following an instructor’s cue; thus promoting creative mathematical activity. This advantage is determinant in the setting of the “regular classroom”, where the characteristics of the students are more diverse than in the MC.

The conditions established for the MC included an institutional setting in which the students worked in an environment propitious for generating creative mathematical activity, and where the roles assigned to the instructor and parents enhanced the conditions in which the didactic design—based on a local praxeology—was implemented. In addition to ensuring the regular participation of their children, the parents participated actively in the MC in the first and last sessions. As Bicknell [47] and Feldhusen [48] proposed, the parents’ participation is an important element in developing mathematical talent. It is important to emphasize that the instructor’s participation in every situation included in the didactic design was fundamental as well, as illustrated by the selection of the sub-task that entailed determining the number of intersections for 20 lines that the instructor proposed to pair B. This allowed those students to recognize that the clearly-stated pattern for the first ten numbers was easily verified for the 20-line case, and so could be generalized to 100 lines. Without doubt, planning these sessions entailed great complexity, for the activities proposed in the didactic design could be performed using diverse techniques and entailed generating proposals for “sub-tasks” that allowed the students to

approach the task situations presented. This leads the authors to recommend elaborating a guide for instructors. In this regard, it is sustained that elaborating a guide for teachers—even holding workshops to professionalize teaching—is necessary for the implementation of such a didactic design. This paper could provide a basis for producing both.

## 6. Conclusions and Future Research

One consequence of this theoretical proposal is that it considers the students in relation to a certain institution through a didactic focus that, instead of emphasizing individual characteristics, analyses how praxeological equipment is provided to develop mathematical talent; specifically, by generating proposals for activities that allow students to construct and reconstruct creative, local-level mathematical praxeologies. This report highlights the solidity of the didactic design for developing the mathematical talent of all the students involved. For example, the students who performed all the tasks successfully (pair C) from situation 2 onwards adapted their techniques to new tasks (F4) and proposed diverse techniques for solving the same task (F3). The students who did not succeed in solving the tasks in situation 2 (pair A) did, however, demonstrate creative mathematical activity in later sessions of the MC, as is clearly evidenced in situation 4.

One of the postulates of this proposal is that creativity is an important element of talent, so developing and analysing it is a fundamental principle for potentiating and observing the evolution of mathematical talent. As a result, the foundation of the PMMT model is creative mathematical activity developed through a solid didactic design in a propitious institutional context. As this creative activity is a necessary, but insufficient, element for realizing the development of mathematical talent, additional elements must be incorporated into the model; for example, considering students' attitudes. This leads to the emergence of another key question for research: How can the PMMT theoretical model be enriched so as to take into account the relations among these additional elements, creative mathematical activity, and the development of mathematical talent?

The didactic design conceived, implemented, and analysed herein constitutes an example of the experimental validation of the PMMT model that, together with the theoretical validation provided by the ATD [49], demonstrates the importance of the model for designing studies of creative mathematical activity, together with its impact on the development of mathematical talent. Nonetheless, it is necessary to implement this model in the context of regular classrooms and/or with various types of tasks.

The PMMT model can be implemented in regular classrooms, but this requires reorganizing study programs to reflect flexible didactic designs based on local mathematical praxeologies. This process demands arduous labors by math teachers due to current institutional conditions (curricular rigidity, evaluation centered on handling algorithms, teaching based on presenting concepts, and administrative obligations, among others), and requires considering students as producers of knowledge. Clearly, this is extremely complex.

**Author Contributions:** Z.M.B.-G., A.R.-V. and S.R.-F. produced the theoretical model presented in this research and supports the didactic design that was implemented in a mathematics club. The execution of the didactic design was in charge of Z.M.B.-G. The three authors, A.R.-V., S.R.-F. and Z.M.B.-G., analysed the transcripts individually and subsequently there was a triangulation, which methodologically supports the analysis and the results shown in this contribution. The three authors wrote this article equally. All authors have read and agreed to the published version of the manuscript.

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